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UNIFORM SEPARATION OF POINTS AND MEASURES AND REPRESENTATION BY SUMS OF ALGEBRAS

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ABSTRACT

Let X and Y_i, $1 \leqq i \leqq k$, be compact metric spaces, and let $\rho_i: X \to Y_i$ be continuous functions. The family $F = \{\rho_i\}_{i=1}^k$ is said to be a *measure separating family* if there exists some $\lambda > 0$ such that for every measure μ in $C(X)^*$, $\|\mu \circ \rho_i^{-1}\| \geqq \lambda \|\mu\|$ holds for some $1 \leqq i \leqq k$. F is a *uniformly (point) separating family* if the above holds for the purely atomic measures in $C(X)^*$. It is known that for $k \leqq 2$ the two concepts are equivalent. In this note we present examples which show that for $k \geqq 3$ measure separation is a stronger property than uniform separation of points, and characterize those uniformly separating families which separate measures. These properties and problems are closely related to the following ones: let $A_1, A_2, \ldots, A_k$ be closed subalgebras of $C(X)$; when is $A_1 + A_2 + \cdots + A_k$ equal to or dense in $C(X)$?

§1. Introduction

Let X and Y_i, $1 \leqq i \leqq k$, be sets, let $\rho_i: X \to Y_i$ be functions, and let λ be a positive real. The family $F = \{\rho_i\}_{i=1}^k$ is said to be a λ-uniformly separating family (λ-u.s.f.) if the following holds: for each pair $\{x_j\}_{j=1}^n$, $\{z_j\}_{j=1}^n$ of finite disjoint sequences in X, there exists some ρ in F so that if from the pair $\{\rho(x_j)\}_{j=1}^n$, $\{\rho(z_j)\}_{j=1}^n$ of sequences, we remove a maximal number of pairs of elements $\rho(x_{j1})$ and $\rho(z_{j2})$ with $\rho(x_{j1}) = \rho(z_{j2})$, at least λn elements will remain in each sequence. (Equivalently: at most $(1-\lambda)n$ pairs can be removed.) F is said to be a u.s.f. if it is a λ-u.s.f. for some $\lambda > 0$.

The following theorem is proved in [4]:

1.1. THEOREM. *The following properties are equivalent*:

(i) *F is a λ-u.s.f.*

(ii) *For each μ in $l_1(X)$ there exists some $1 \leqq i \leqq K$ such that $\|\mu \circ \rho_i^{-1}\| \geqq \lambda \|\mu\|$.*

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(iii) *Each f in $l_\infty(X)$ admits a representation $f(x)=\Sigma_{i=1}^k g_i(\rho_i(x))$ with g_i in $l_\infty(Y_i)$, $\|g_i\| \leqq (1/\lambda)\|f\|$ ($l_1(X)$ is the Banach space of purely atomic measures on X with the total variation as the norm; $\mu \circ \rho_i^{-1}$ is the measure on Y_i defined by $\mu \circ \rho_i^{-1}(A)=\mu(\rho_i^{-1}(A))$, $A \subset Y_i$; $l_\infty(X)$ is the space of bounded real valued functions on X with the sup norm).*

If X and Y_i are compact metric spaces, and the ρ_i's are continuous, then F is said to be a λ-measure separating family if for each real valued Borel measure μ in $C(X)^*$ there is some ρ_i in F so that $\|\mu \circ \rho_i^{-1}\| \geqq \lambda \|\mu\|$.

As in the discrete case we have (see [4])

1.2. THEOREM. *F is a λ-measure separating family if and only if each f in $C(X)$ admits a representation $f(x)=\Sigma_{i=1}^k g_i(\rho_i(x))$ with g_i in $C(Y_i)$, $\|g_i\| \leqq (1/\lambda)\|f\|$.*

It follows that every measure separating family is a u.s.f. (with the same λ). The main purpose of this paper is to study the inverse problem, i.e., when is a u.s.f. also measure separating? Obviously, if $k=1$ (i.e., F consists of a single element) then the two properties are equivalent. In [4], p. 69, it has been proven that if $k \leqq 2$ then, again, the two properties are equivalent, i.e., if $F=\{\rho_1,\rho_2\}$ is a u.s.f. then it is measure separating. In this article we present an example of a family $F=\{\rho_1,\rho_2,\rho_3\}$ of continuous functions on some compact metric space X, which is a $\frac{1}{3}$-u.s.f., but fails to separate measures in the strongest possible sense: there exists a Borel measure μ in $C(X)^*$ with $\|\mu\|=1$ and $\mu \circ \rho_i^{-1}=0$ for $i=1,2,3$. Thus, for $k>2$, the two properties are no longer equivalent. We were able to construct this example after reading Marshall and O'Farrell's paper [2] and, in particular, the example due to Havinson which is presented there. In that article the authors study the following problem: let $A_1, A_2, \ldots, A_k$ be closed subalgebras of $C(X)$ (with $1 \in A_i$, $1 \leqq i \leqq k$), when is $A_1+A_2+\cdots+A_k$ dense in $C(X)$? Clearly, every such A_i is of the form $A_i=C(Y_i)$ with Y_i a quotient of X. It follows from 1.2 that $A_1+A_2+\cdots+A_k=C(X)$ if and only if the family $F=\{\rho_1,\rho_2,\ldots,\rho_k\}$ separates measures, where $\rho_i: X \to Y_i$ is the quotient map. It is also evident that $A_1+\cdots+A_k$ is dense in $C(X)$ if and only if, for all $\mu \neq 0$ in $C(X)^*$, $\mu \circ \rho_i^{-1} \neq 0$ for some $1 \leqq i \leqq k$. From the results of Marshall and O'Farrell it follows, in particular, that when $k=2$, every element μ in $(A_1+A_2)^\perp$ is a w* limit of a sequence $\{\mu_n\}_{n=1}^\infty$ in $l_1(X)$ such that each μ_n has a finite support, and

$$\lim_{n\to\infty} (\|\mu_n \circ \rho_1^{-1}\| + \|\mu_n \circ \rho_2^{-1}\|) = 0.$$

From our example it follows that similar results cannot be obtained for $k > 2$. Indeed, the element μ in our example is in $(A_1 + A_2 + A_3)^\perp$, but for any sequence $\{\mu_n\}$ in $l_1(X)$ which tends w* to μ,

$$\lim_{n\to\infty} \sup(\|\mu_n \circ \rho_1^{-1}\| + \|\mu_n \circ \rho_2^{-1}\| + \|\mu_n \circ \rho_3^{-1}\|) \geqq \tfrac{1}{3}$$

holds. (Actually the lim sup is $\geqq 1$, as will be seen later.) We present our example, in a general setting, in §2. In §3, we examine special cases with some additional properties, one with $X \subset R^2$ and real valued ρ_i's, and another in which X is connected.

Once we know that a u.s.f. may fail to separate measures, it is of some interest to characterize those u.s.f.'s which do. This is done in §4, where we present a necessary and sufficient condition on a u.s.f. in order to be a measure separating family. We hope that it will be possible to apply this characterization to extend the results of Marshall and O'Farrell to the case $k \geqq 3$ in some meaningful way.

§2. A u.s.f. which does not separate measures

Let G be a compact metrizable uncountable topological group which contains two elements a and b that generate a non-abelian free group on two generators. Examples of such groups will be presented in §3.

Let X be the disjoint union of two copies G_0 and G_1 of G. Define ρ_i: $X \to G$, $i = 1,2,3$, as follows:

$$\rho_1(x) = \begin{cases} x, & x \in G_0, \\ xa, & x \in G_1, \end{cases} \qquad \rho_2(x) = \begin{cases} xb, & x \in G_0, \\ x, & x \in G_1, \end{cases} \qquad \rho_3(x) = \begin{cases} x, & x \in G_0 \\ x, & x \in G_1 \end{cases}$$

(xa is the product of x and a, in the given order, in the group G). Set $F = \{\rho_1, \rho_2, \rho_3\}$. We claim that F is a $\frac{1}{3}$-u.s.f. on X but fails to be a measure separating family. The last statement is obvious: let ν denote the Haar measure on G, and set $\mu = \frac{1}{2}(\nu_0 - \nu_1)$ where ν_i is ν on G_i, $i = 0,1$. Then $\|\mu\| = 1$, and $\mu \circ \rho_i^{-1} = 0$ for $i = 1,2,3$, since ν is translation invariant. To prove that F_3^1 is a $\frac{1}{3}$-u.s.f. we argue as follows. Let $H \subset G$ be the non-abelian free group generated by a and b. For $w \in G$ let wH denote the coset $\{wu: u \in H\}$. These cosets form a decomposition of G into disjoint sets. They also induce naturally such a decomposition on X. Indeed, for $w \in G$ set

$$x_w = wH_0 \cup wH_1, \qquad \text{where } wH_i \text{ is the coset } wH \text{ in } G_i, \quad i = 0,1.$$

This decomposition of X is respected by the three ρ_i's, $i = 1,2,3$, i.e., if X_u and

X_w, $u, w \in G$, are two different elements in the decomposition of X, then $\rho_i(X_u) \cap \rho_i(X_w) = \emptyset$, since ρ_i maps X_w into wH, $i = 1, 2, 3$.

It follows that in order to prove that F is a $\frac{1}{3}$-u.s.f., it suffices to show that the restriction of the elements of F to X_w for $w \in G$ satisfy the following:

2.1. For every $\eta \in l_1(X_w)$, $\Sigma_{i=1}^3 \| \eta \circ (\rho_i / X_w)^{-1} \| \geqq \| \eta \|$.

(Note that 2.1 implies that F/X_w is a $\frac{1}{3}$-u.s.f. on X_w, since from 2.1 it follows that $\| \eta \circ (\rho_i / X_w)^{-1} \| \geqq \frac{1}{3} \| \eta \|$ for some $1 \leqq i \leqq 3$.)

Indeed, given some η in $l_1(X)$, there exists a sequence $\{w_k\}_{k=1}^\infty$ in G such that $\bigcup_{k=1}^\infty X_{w_k}$ contains the support of η (since the support of η is countable). Hence $\eta = \Sigma_{k=1}^\infty \eta / X_{w_k}$. From 2.1 and from the fact that the ρ_i's map disjoint X_w's into disjoint sets, it follows that

2.2.

$$\sum_{i=1}^3 \| \eta \circ \rho_i^{-1} \| = \sum_{i=1}^3 \left\| \sum_{k=1}^\infty \eta / X_{w_k} \circ (\rho_i / X_{w_k})^{-1} \right\| = \sum_{i=1}^3 \sum_{k=1}^\infty \| \eta / X_{w_k} \circ (\rho_i / X_{w_k})^{-1} \|$$
$$= \sum_{k=1}^\infty \sum_{i=1}^3 \| \eta / X_{w_k} \circ (\rho_i / X_{w_k})^{-1} \| \geqq \sum_{k=1}^\infty \| \eta / X_{w_k} \| = \| \eta \|.$$

From 2.2 it now follows that for some $1 \leqq i \leqq 3$, $\| \eta \circ \rho_i^{-1} \| \geqq \frac{1}{3} \| \eta \|$, i.e., F is $\frac{1}{3}$-u.s.f. Thus 2.1 actually implies a stronger property, namely 2.2, than just being a $\frac{1}{3}$-u.s.f. (This also explains the remark about the lim sup being $\geqq 1$ in the introduction.)

To prove 2.1 we shall need a lemma.

2.3. Lemma. *Let $f = \{\rho_i\}_{i=1}^k$ be a family of functions on a set X. If there exist subsets $A_1, A_2, \ldots, A_k$ of X, and r positive, such that*

(i) *for each x in X, $\Sigma_{i=1}^k 1_{A_i}(x) \geqq r$ (i.e., each x in X is an element of at least r A_i's).*

(ii) *ρ_i separates the points of A_i, $1 \leqq i \leqq k$ (i.e., ρ_i / A_i is one to one on A_i), then for each $\eta \in l_1(X)$, $\Sigma_{i=1}^k \| \eta \circ \rho_i^{-1} \| \geqq (2r - k) \| \eta \|$.*

Proof. Let $\eta \in l_1(X)$ with $\| \eta \| = 1$ be given. Then $|\eta|$, the absolute value of η, is a probability measure on X. From (i) it then follows that

2.4.

$$\sum_{i=1}^k |\eta|(A_i) = \sum_{i=1}^k \int 1_{A_i}(x) d|\eta|(x) = \int \sum_{i=1}^k 1_{A_i}(x) d|\eta|(x) \geqq \int r d|\eta|(x) = r.$$

Let $A_i^c = X \setminus A_i$ be the complement of A_i. Then

2.5.

$$\sum_{i=1}^k |\eta|(A_i^c) \leqq k - r.$$

For a fixed i, ρ_i is one to one on A_i, hence $\|\eta/A_i \circ (\rho_i/A_i)^{-1}\| = \|\eta/A_i\| = |\eta|(A_i)$. The part of η which lies out of A_i, in A_i^c, may reduce this quantity by at most $\|\eta/A_i^c\|$. Thus we have

$$\|\eta \circ \rho_i^{-1}\| \geqq \|\eta/A_i \circ (\rho_i/A_i)^{-1}\| - \|\eta/A_i^c\| \geqq |\eta|(A_i) - |\eta|(A_i^c).$$

Hence

$$\sum_{i=1}^{k} \|\eta \circ \rho_i^{-1}\| \geqq \sum_{i=1}^{k} \big|\, |\eta|(A_i) - |\eta|(A_i^c)\big| \geqq r - (k-r) = 2r - k,$$

and the lemma follows.

In particular, for $k = 3$ and $r = 2$ we obtain

2.6.
$$\sum_{i=1}^{3} \|\eta \circ \rho_i^{-1}\| \geqq 4 - 3 = 1.$$

Hence, in order to prove 2.1 it suffices to show the existence of subsets A_1, A_2, and A_3 of X_w so that (i) and (ii) of Lemma 2.3 hold with $r = 2$. Let e denote the unit element of H (and of G). We shall present the sets A_i explicitly for $X_e = H_0 \cup H_1$, and for $X_w = wH_0 \cup wH_1$, the corresponding sets will simply be wA_i, $i = 1, 2, 3$.

Being a non-abelian free group on the two generators a and b, each $x \in H$, $x \neq e$ admits a unique representation as a reduced "word" in the symbols a, b, a^{-1}, b^{-1}. For $x \in H$ let $r(x)$ denote the symbol which appears in the right-hand side of the reduced word which represents x, and $r(x) = e$ if $x = e$. (For example, $r(a\,a\,b\,a^{-1}\,b^{-1}\,a\,b^{-1}) = b^{-1}$.) Set

$$A_1 = \{x \in H_0\colon r(x) \neq a\} \cup \{x \in H_1\colon r(x) \neq a^{-1}\},$$

$$A_2 = \{x \in H_0\colon r(x) \neq b^{-1}\} \cup \{x \in H_1\colon r(x) \neq b\},$$

$$A_3 = \{x \in H_0\colon r(x) \in \{a, b^{-1}\}\} \cup \{x \in H_1\colon r(x) \in \{a^{-1}, b\}\},$$

ρ_i is one to one on A_i: since each ρ_i separates the points of both G_0 and G_1, one has only to show that if $x \in A_i \cap G_0$ and $y \in A_i \cap G_1$, then $\rho_i(x) \neq \rho_i(y)$. For $i = 1$, recall that

$$\rho_1(x) = \begin{cases} x, & x \in G_0 \\ xa, & x \in G_1 \end{cases}$$

so, if $x \in A_1 \cap G_0$, then $\rho_1(x) = x$ and $r(x) = r(\rho_1(x)) \neq a$. If $y \in A_1 \cap G_1$, then $r(y) \neq a^{-1}$, and $\rho_1(y) = ya$. As $r(y) \neq a^{-1}$, ya is actually the reduced representation of $\rho_1(y)$ (provided y is assumed to be a reduced word) thus $r(\rho_1(y)) = a$, and since $r(\rho_1(x)) \neq a$ it follows that $\rho_1(x) \neq \rho_1(y)$. A similar argument shows that ρ_2 separates the points of A_2. Recall that ρ_3 acts as the identity on both G_0 and G_1, thus, since if $x \in A_3 \cap G_1$ and $y \in A_3 \cap G_2$, x and y are not the same element of G, it follows that $\rho_3(x) \neq \rho_3(y)$. Hence (i) of 2.3 holds. (ii) of 2.3 holds too. As a matter of fact, $\Sigma_{i=1}^3 1_{A_i}(x) = 2$ for all $x \in X_e$. This is checked easily by considering the cases $x \in H_0$ and $x \in H_1$, and $r(x) = e, a, b, a^{-1}, b^{-1}$, separately. The details are left to the reader. This accomplishes our construction.

Note that Lemma 2.3 applies to $C(X)^*$ as well as to $l_1(X)$ if the A_i's are Borel sets. The sets $B_i = \bigcup_w wA_i$, $i = 1, 2, 3$ (one w for each coset of H) satisfy (i) and (ii) of 2.3 but are not Borel sets. On the other hand, $\eta(X_w) = 0$ for every atomless η. Note also that if H is dense in G then μ is the only (up to a real factor) element of $C(X)^*$ which is annihilated by the three ρ_i's.

§3. Some special cases

The space X in §2 is not connected and the maps ρ_i are not real valued. In this section we present a connected example, and also show that every finite dimensional example can be modified in order to obtain one with real valued ρ_i's. But first we present a zero-dimensional $X \subset R^2$ with three real valued ρ_i's, two of which being the coordinate projections. I am indebted to A. O'Farrell for bringing this example to my attention.

Let p be a fixed prime, and let Z_p denote the compact, zero-dimensional, uncountable metrizable ring of p-adic integers. (See [1], pp. 85–94.) The group $G = \mathrm{SL}(2, Z_p)$ of 2×2 matrices with determinant one over Z_p is a compact metrizable zero-dimensional uncountable group; moreover, it contains a free group on two generators. Indeed, the elements

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate such a group. (See [6], p. 97.) being zero-dimensional compact metric spaces, both $X = G_0 \cup G_1$ and G can be considered as a subset of the real line R. Moreover, since the (real valued) functions ρ_i, $i = 1, 2$, separate the points of X, we can embed X in R^2 by $x \to (\rho_1(x), \rho_2(x))$. We have thus verified the following

PROPOSITION. *There exists a compact subset X of R^2, and an element ρ_3 of $C(X)$ such that $\rho_1(x, y) = x$, $\rho_2(x, y) = y$ and ρ_3 form a $\frac{1}{3}$-u.s.f. on X which fails to separate measures. Dually stated: every bounded real-valued function on X admits a representation*

$$f(x, y) = g_1(x) + g_2(y) + g_3(\rho_3(x, y)), \qquad (x, y) \in X$$

with g_i bounded real-valued on R, while $\{g_1(x)+g_2(y)+g_3(\rho_3(x,y)): g_i \in C(R)\}$ is not even dense in $C(X)$.

The group SO_3 of orthogonal transformations on R^3 also contains a free group on two generators. (See [6], p. 15 for a short proof.) $G = SO_3$ is also connected and 3-dimensional. We apply it to construct a connected example with real-valued functions.

Let e denote the unit element of G, let G_0 and G_1 be two disjoint copies of G, and let e_0 and e_1 denote the element e in G_0 and G_1 respectively. Let $Z = G_0 \cup [0,1] \cup G_1$ be the union of G_0, the unit interval $[0,1]$ and G_1, with the following identification of two pair of points: e_0 in G_0 is identified with $0 \in [0,1]$, and $e_1 \in G_1$ is identified with $1 \in [0,1]$. With the exception of these two identifications, the union is disjoint. It follows that if G is connected then so is Z.

The space X described in §2 can be looked upon as a closed subset of Z. Let $Y_1 = G \cup [0,1]$ denote the union of G with $[0,1]$ with the following identifications: $e \in G$ and $0 \in [0,1]$ are identified, and so are $a \in G$ and $1 \in [0,1]$. The following map ψ_1: $Z \to Y_1$ is an extension of ρ_1: $X \to G$,

$$\psi_1(x) = \begin{cases} x, & x \in G_0, \\ x, & x \in [0,1], \\ xa, & x \in G_1. \end{cases}$$

Note that ψ_1 is continuous, since the identified pair $(e_0, 0)$ in Z is mapped to the identified pair $(e, 0)$ in Y_1, and the identified pair $(e_1, 1)$ in Z is mapped to the identified pair $(a, 1)$ in Y_1.

Note also that the open interval $(0,1)$ in Z is mapped homeomorphically by ψ_1 onto the corresponding interval in Y_1, and that no point of $X = G_0 \cup G_1 \subset Z$ is mapped into this interval, i.e., $\psi_1^{-1}(Y_1 \setminus G) = Z \setminus X$.

Similarly, let Y_2 be the union of G and $[0,1]$ with the following identifications: $b \in G$ and $0 \in [0,1]$ are identified, and so are $e \in G$ and $1 \in [0,1]$. ψ_2: $Z \to Y_2$ is then a continuous extension of ρ_2 when defined as follows:

$$\psi_2(x) = \begin{cases} xb, & x \in G_0, \\ x, & x \in [0,1], \\ x, & x \in G_1. \end{cases}$$

The above remark about the behaviour of ψ_1 on $(0,1) \subset Z$ applies also to ψ_2.

Finally, set $Y_3 = G$, and extend ρ_3 to a map ψ_3: $Z \to Y_3$ by letting

$$\psi_3(x) = \begin{cases} x, & x \in G_0, \\ e, & x \in [0,1], \\ x, & x \in G_1. \end{cases}$$

We claim that $F = \{\psi_1, \psi_2, \psi_3\}$ is a $\frac{1}{3}$-u.s.f. on Z, and that $\mu \circ \psi_i^{-1} = 0$ for $i = 1,2,3$, where μ is the measure defined in §2. The second statement being trivial, we proceed to prove the first. The following is a decomposition of Z into disjoint sets: $Z_e = X_e \cup (0,1)$, $Z_w = X_w$, $w \in G \setminus H$. As in §2 it suffices to show that 2.3 (with $k = 3$ and $r = 2$) can be applied to each Z_w. For Z_w, $w \in G \setminus H$, we can simply adopt the sets wA_1, wA_2 and wA_3 from §2. For Z_e, we take the sets $B_1 = A_1 \cup (0,1)$, $B_2 = A_2 \cup (0,1)$ and $B_3 = A_3$. It is evident that each x in Z_e appears as an element in exactly two B_i's: if $x \in (0,1)$ then $x \in B_1$ and $x \in B_2$, and if $x \in X_e$ then it follows from the properties of A_i, $i = 1,2,3$. Also, ψ_i is one to one on B_i: this is trivial for $i = 3$, and follows from the fact that ψ_i is an extension of ρ_i, and the remarks following the definitions of ψ_i, $i = 1,2$. This proves that F is a $\frac{1}{3}$-u.s.f. on Z.

Let $\dim G$ denote the topological dimension of G. If $\dim G$ is finite, then $\dim Y_i$ is finite too. (Recall that $\dim SO_3 = 3$.) By a theorem of Ostrand ([3], see also [5] for a proof), there exists, for each $1 \leqq i \leqq 3$, a $(2\dim G + 1)^{-1}$ measure separating family $F_i = \{\tau_{i,j}\}_{j=1}^{2\dim G+1}$ on Y_i, with $F_i \subset C(Y_i)$. The family $E = \{\psi_i \circ \tau_{i,j}\}$, $1 \leqq i \leqq 3$, $1 \leqq j \leqq 2\dim G + 1$, is then a $(3 \cdot (2\dim G + 1))^{-1}$ u.s.f. on Z, with Z connected and real-valued functions $\psi_i \circ \tau_{i,j}$; and clearly the above-mentioned measure μ will be annihilated by all the mappings $\psi_i \circ \tau_{i,j}$.

§4. A necessary and sufficient condition for a u.s.f. to be measure separating

The property of being a u.s.f. is essentially a combinatorial one, while measure separation, in addition to being a combinatorial conditions, is also related to the Borel structure, and hence to the topological structure of the spaces involved. Hence it is not surprising that a u.s.f. may fail to be a measure separating family. The fact that for $k = 2$ the two properties are equivalent should be considered to be a surprising one. It is therefore also natural to expect that the condition which characterizes measure separating families in terms of uniform separation of points, will combine the combinatorial properties of a u.s.f. with the topology of the corresponding spaces. Before we present it we wish to obtain some better understanding of the combinatorial and probabilistic nature of separation of points and of measures.

4.1. LEMMA. *Let X and Y be measurable spaces, let $\rho: X \to Y$ be a measurable function, let μ be a real-valued measure on X, with $\|\mu\| = 1$, and let $0 < \lambda \leqq 1$. The following are then equivalent*:

(i) $\|\mu \circ \rho^{-1}\| \geqq \lambda$.

(ii) *If $Y = Y^+ \cup Y^-$ is the (some) Hahn decomposition of Y with respect to the* *measure* $\mu \circ \rho^{-1}$, *then* $\mu^+(\rho^{-1}(Y^+)) + \mu^-(\rho^{-1}(Y^-)) \geqq (1+\lambda)/2$, *where* μ^+ *and* μ^- *are the positive and negative parts of* μ *repectively.*

(iii) *Let* $X = X^+ \cup X^-$ *be the Hahn decomposition of* X *w.r.t. the measure* μ. *There exist subsets* U^+ *of* X^+ *and* U^- *of* X^- *such that* $\mu^+(U^+) + \mu^-(U^-) = \mu(U^+) + |\mu(U^-)| \geqq (1+\lambda)/2$ *and* $\rho(U^+) \cap \rho(U^-) = \varnothing$.

(iv) *There exists a subset* V *of* X *such that* $\|\mu/V\| = \|\mu/V \circ (\rho/V)^{-1}\| \geqq (1+\lambda)/2$.

PROOF. (i)→(ii). Assume that $\|\mu \circ \rho^{-1}\| \geqq \lambda$. Then

(a) $1 = |\mu|(X) = |\mu|(\rho^{-1}(Y^+)) + |\mu|(\rho^{-1}(Y^-))$

$$= \mu^+(\rho^{-1}(Y^+)) + \mu^-(\rho^{-1}(Y^+)) + \mu^+(\rho^{-1}(Y^-) + \mu^-(\rho^{-1}(Y^-))$$

(where $|\mu| = \mu^+ + \mu^-$, $\mu = \mu^+ - \mu^-$, $\mu^\pm = \pm\mu/X^\pm$).
And since $Y = Y^+ \cup Y^-$ is the Hahn decomposition of Y w.r.t. $\mu \circ \rho^{-1}$ we also have:

(b) $\lambda \leqq \|\mu \circ \rho^{-1}\| = |\mu \circ \rho^{-1}|(Y) = |\mu \circ \rho^{-1}|(Y^+) + |\mu \circ \rho^{-1}|(Y^-)$

$$= \mu \circ \rho^{-1}(Y^+) - \mu \circ \rho^{-1}(Y^-) = \mu(\rho^{-1}(Y^+)) - \mu(\rho^{-1}(Y^-))$$

$$= \mu^+(\rho^{-1}(Y^+)) - \mu^-(\rho^{-1}(Y^+)) - (\mu^+(\rho^{-1}(Y^-)) + \mu^-(\rho^{-1}(Y^-))).$$

Summing (a) and (b) we obtain $2\mu^+(\rho^{-1}(Y^+)) + 2\mu^-(\rho^{-1}(Y^-)) \geqq 1 + \lambda$, and (ii) follows.

(ii)→(iii). Assume (ii), and set $U^\pm = X^\pm \cap \rho^{-1}(Y^\pm)$. Then

$$\mu^+(U^+) + \mu^-(U^-) = \mu^+(\rho^{-1}(Y^+)) + \mu^-(\rho^{-1}(Y^-)) \geqq (1+\lambda)/2$$

and

$$\rho(U^+) \cap \rho(U^-) = \varnothing$$

since $\rho(U^\pm) \subset Y^\pm$.

(iii)→(iv). Assume (iii) and set $V = U^+ \cup U^-$. It is easy to check that (iv) holds.

(iv)→(i). Let V satisfy (iv). Then clearly $\|\mu/X \setminus V\| \leqq 1 - (1+\lambda)/2 = (1-\lambda)/2$. Hence

$$\|\mu \circ \rho^{-1}\| \geqq \|\mu/V \circ (\rho/V)^{-1}\| - \|\mu/X \setminus V\| \geqq (1+\lambda)/2 - (1-\lambda)/2 = \lambda,$$

and the lemma is proved.

The following proposition follows easily from 1.1 and 4.1.

4.2. PROPOSITION. *F is a λ-u.s.f. on X if and only if the following holds: for every pair $A=\{x_j\}_{j=1}^n$ and $B=\{z_j\}_{j=1}^k$ of finite disjoint sequences in X with length $|A|=n$ and $|B|=k$, there exists some ρ in F and subsequences A' of A and B' of B so that $|A'|+|B'|\geqq(|A|+|B|)(1+\lambda)/2$ and $\rho(A')\cap\rho(B')=\emptyset$.*

PROOF. Let F be a λ-u.s.f., and let A and B be two sequences as above. Consider the element $\mu=\Sigma_{j=1}^n\delta_{x_j}-\Sigma_{j=1}^k\delta_{z_j}$ of $l_1(X)$ (where δ_x is the Dirac measure). Then $\|\mu\|=n+k$, and by (ii) of 1.1 there exists some ρ in F so that $\|\mu\circ\rho^{-1}\|\geqq\lambda\|\mu\|$. By (iii) of 4.1 there exist subsets U^+ of A and U^- of B with $\mu^+(U^+)+\mu^-(U^-)\geqq(n+k)(1+\lambda)/2$ and $\rho(U^+)\cap\rho(U^-)=\emptyset$. Set $A'=U^+$ and $B'=U^-$, and we are done.

Conversely, let $A=\{x_j\}_{j=1}^n$ and $B=\{z_j\}_{j=1}^n$ be disjoint sequences in X. Set $\mu=\Sigma_{j=1}^n\delta_{x_j}-\Sigma_{j=1}^n\delta_{z_j}$; μ is an element of $l_1(X)$ with $\|\mu\|=2n$. Let $\rho\in F$ and $A'\subset A$, $B'\subset B$ be such that $|A'|+|B'|\geqq n(1+\lambda)$ and $\rho(A')\cap\rho(B')=\emptyset$. By (iii)$\rightarrow$(i) of 4.1 it follows that $\|\mu\circ\rho^{-1}\|\geqq 2\lambda n$. But from this it follows that in the process of removing pairs $\rho(x_{j_2})=\rho(z_{j_2})$, at least λ_n pairs must remain, since otherwise we would have $\|\mu\circ\rho^{-1}\|<2\lambda n$. This proves the proposition.

It is the equivalent version of 4.2 which we use to define our condition.

4.3. DEFINITION. Let (X,d) and (Y_i,d_i), $1\leqq i\leqq k$, be compact metric spaces, and let $F=\{\rho_i\}_{i=1}^k$ where $\rho_i\colon X\rightarrow Y_i$ are continuous functions. Let λ be a positive real. F is said to be a λ-uniformly u.s.f. (λ-u^2.s.f.) if the following holds:

For each $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ so that for every pair A, B of finite sequences in X with $d(A,B)\geqq\varepsilon$, there exists some ρ_i in F and subsequences A' of A and B' of B with

$$|A'|+|B'|\geqq(|A|+|B|)(1+\lambda)/2 \quad\text{and}\quad d_i(\rho_i(A'),\rho_i(B'))\geqq\delta.$$

Note that if F is merely a λ-u.s.f., then given disjoint sequences A and B in X, a $\delta>0$ which satisfies the above always exists, but in general δ will depend on the sequences A and B and not only on the distance between them.

4.4. THEOREM. *Let X, Y_i, $1\leqq i\leqq k$, F and λ be as in 4.3. Then F is a measure separating family if and only if it is a u^2.s.f. More precisely: if F is a λ-u^2.s.f., then it is a λ-measure separating family, while if F is a λ-measure separating family, then for every $\lambda'<\lambda$ it is a λ'-u^2.s.f.*

PROOF. Let F be a λ-u^2.s.f. Let $\mu\in C(X)^*$ with $\|\mu\|=1$ be given. Let S^+ and S^- denote the support of μ^+ and μ^- respectively. Without loss of generality we may assume that the compact sets S^+ and S^- are disjoint. Indeed, measures μ

with this property are dense in $C(X)^*$, and since $\mu \to \mu \circ \rho_i$ is a bounded linear operator it suffices to prove that $\max_{1\leq i\leq k} \|\mu \circ \rho_i^{-1}\| \geqq \lambda$ for such measures only. Set $\varepsilon = d(S^+, S^-)$, and let $\delta > 0$ be the one from 4.3. For $l = 1, 2, \dots$ let

$$\mu_l = \sum_{j=1}^{U_l} r_j^l \delta_{x_j^l} - \sum_{j=1}^{V_l} s_j^l \delta_{z_j^l}$$

be elements of $l_1(X)$ so that

(i) $\{x_j^l\}_{j=1,l=1}^{U_l,\infty} \subset S^+$, $\{z_j^l\}_{j=1,l=1}^{V_l,\infty} \subset S^-$,

(ii) μ_l tends w* to μ,

(iii) $r_j^l = a_j^l/n_l$ and $s_j^l = b_j^l/k_l$ are positive rationals with n_l a common denominator for $r_1^l, \dots, r_{U_l}^l$, and k_l a common denominator for $s_1^l, \dots, s_{V_l}^l$, and $\|\mu^+\| = \sum_{j=1}^{U_l} r_j^l$, $\|\mu^-\| = \sum_{j=1}^{U_l} s_j^l$ for $l = 1, 2, \dots$. (We assume also that $\|\mu^+\|$ and $\|\mu^-\|$ are rationals.)

For $l \geqq 1$, let A_l be a sequence in S^+ whose elements are the x_j^l's, and in which each x_j^l appears a_j^l times as an element. Then the length of A_l is

$$|A_l| = \sum_{j=1}^{U_l} a_j^l = n_l \|\mu^+\|.$$

Similarly, let $B_l \subset S^-$ be a sequence whose elements are the s_j^l's, each appearing b_j^l times, with

$$|B_l| = k_l \|\mu^-\| = \sum_{j=1}^{V_l} b_j^l.$$

It follows that $d(A_l, B_l) \geqq \varepsilon$. From F being a λ-u^2.s.f. it follows that for each $l \geqq 1$ there exists some $\rho_i \in F$ and subsequences $A_l' \subset A_l$ and $B_l' \subset B_l$, with $|A_l'| + |B_l'| \geqq (n_l \|\mu^+\| + k_l \|\mu^-\|)\,(1+\lambda)/2$ and $d_i(\rho_i(A_l'), \rho_i(B_l')) \geqq \delta$. By passing to a subsequence of the l's, we may assume that the same $\rho_i = \rho$ does the job for all the l's. Let $V^+ \subset Y_i$ be the limit in the Hausdorff metric on the closed subsets of Y_i, of a subsequence of the $\rho(A_l')$'s. Let $V^- \subset Y_i$ be the limit in the same metric of a further subsequence of the $\rho(B_l')$'s. Then $d_i(V^+, V^-) \geqq \delta$, and hence there exist disjoint open subsets $U^\pm$ of Y_i so that $V^\pm \subset U^\pm$. It follows that for infinitely many l's, and without loss of generality for all l, $\rho(A_l') \subset U^+$ and $\rho(B_l') \subset U^-$.

Hence, for all l, $A_l' \subset S^+ \cap \rho^{-1}(U^+)$ and $B_l' \subset S^- \cap \rho^{-1}(U^-)$. Thus

$$\mu_l^+(\rho^{-1}(U^+)) \geqq |A_l'|/n_l \geqq \|\mu^+\|(1+\lambda)/2$$

and also

$$\mu_l^-(\rho^{-1}(U^-)) \geqq |B_l'|/k_l \geqq \|\mu^-\|(1+\lambda)/2.$$

From the fact that $\mu_l^{\pm} \xrightarrow{w^*} \mu^{\pm}$ it follows that

$$\mu^+(\rho^{-1}(U^+)) \geqq \|\mu^+\|(1+\lambda)/2 \quad \text{and} \quad \mu^-(\rho^{-1}(U^-)) \geqq \|\mu^-\|(1+\lambda)/2.$$

Hence, $\mu^+(\rho^{-1}(U^+)) + \mu^-(\rho^{-1}(U^-)) \geqq (\|\mu^+\| + \|\mu^-\|)(1+\lambda)/2 = (1+\lambda)/2$ and, by (iii)→(i) of 4.1, $\|\mu \circ \rho^{-1}\| \geqq \lambda$, i.e., F is a λ-measure separating family.

Conversely, assume that F is a λ-measure separating family, let $\lambda' < \lambda$ be given, and assume that F fails to be a λ'-u^2.s.f. Hence, for some $\varepsilon > 0$, there exist for each $l \geqq 1$ a pair A_l, B_l of finite sequences in X with $d(A_l, B_l) \geqq \varepsilon$, so that for any choice of $A'_l \subset A_l$ and $B'_l \subset B_l$ with $|A'_l| + |B'_l| \geqq (|A_l| + |B_l|)(1+\lambda')/2$,

$$\lim_{l \to \infty} d_i(\rho_i(A'_l), \rho_i(B'_l)) = 0$$

holds for all $1 \leqq i \leqq k$. Set

$$\mu_l = \frac{1}{|A_l| + |B_l|}\left(\sum_{x \in A_l} \delta_x - \sum_{z \in B_l} \delta_z\right).$$

Then $\mu_l \in l_1(X)$ and $\|\mu_l\| = 1$.

By passing to a subsequence if necessary we may assume that $\{\mu_l\}_{l=1}^{\infty}$ converges w* to some element μ of $C(X)^*$. From $d(A_l, B_l) \geqq \varepsilon$ it follows that $\|\mu\| = 1$. Since F is a λ-measure separating, there exists some $\rho_i = \rho$ in F so that $\|\mu \circ \rho^{-1}\| \geqq \lambda$. By (ii) of 4.1, $\mu^+(\rho^{-1}(Y^+)) + \mu^-(\rho^{-1}(Y^-)) \geqq (1+\lambda)/2$ where $Y^+ \cup Y^-$ is the Hahn decomposition of $Y = Y_i$ w.r.t. $\mu \circ \rho^{-1}$. By the regularity of the measures $\mu^{\pm} \circ \rho^{-1}$ we can find compact subsets $V^{\pm}$ of $Y^{\pm}$ so that $\mu^+(\rho^{-1}(V^+)) + \mu^-(\rho^{-1}(V^-)) \geqq (1+\lambda'')/2$, with $\lambda' < \lambda'' < \lambda$.

Let $U^{\pm}$ be open subsets of Y with $V^{\pm} \subset U^{\pm}$ and $d_i(U^+, U^-) = \delta > 0$. Then clearly also $\mu^+(\rho^{-1}(U^+)) + \mu^-(\rho^{-1}(U^-)) \geqq (1+\lambda'')/2$. It follows that for all sufficiently large l, $\mu_l^+(\rho^{-1}(U^+)) + \mu_l^-(\rho^{-1}(U^-)) \geqq (1+\lambda')/2$ (since $\rho^{-1}(U^{\pm})$ is open in X). This implies that $A'_l = A_l \cap \rho^{-1}(U^+)$ and $B'_l = B_l \cap \rho^{-1}(U^-)$ satisfy $|A'_l| + |B'_l| \geqq (|A_l| + |B_l|)(1+\lambda')/2$ while

$$d_i(\rho(A'_l), \rho(B'_l)) \geqq d_i(U^+, U^-) = \delta > 0.$$

This contradiction proves that F must be a λ'-u^2.s.f., and the theorem is proved.

Note that from Theorem 4.4, and the fact that when F is a u.s.f. with only one or two elements in it then F is measure separating, it follows that every such u.s.f. is actually a u^2.s.f.

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